

Integral manifolds of the reduced system in the problem of inertial motion of a rigid body about a fixed point*

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Let A , B and C ($A < B < C$) be the principal moments of inertia of a rigid body having a fixed point. We consider the ellipsoid E^2 defined in \mathbb{R}^3 by the equation $Ax^2 + By^2 + Cz^2 = 1$ with the following metric on its surface

$$d\Sigma = \sqrt{hABC} (A^2x^2 + B^2y^2 + C^2z^2)^{-\frac{1}{2}} d\sigma.$$

Here $h > 0$ is a constant, $d\sigma$ is the metric on E^2 induced by the scalar product of \mathbb{R}^3 . The reduced system [1] of the problem of the motion of a rigid body with a fixed point without external forces is equivalent to the Hamiltonian system on T^*E^2 with the Hamilton function

$$H = \frac{A^2x^2 + B^2y^2 + C^2z^2}{2ABC} (p_x^2 + p_y^2 + p_z^2). \quad (1)$$

The projections to E^2 of the integral curves of this system with constant energy $H = h$ are, according to the Maupertuis principle, geodesics of the metric $d\Sigma$. Thus, instead of saying “the basic integral curve of the Hamiltonian vector field X_H ” we use the term “geodesic”.

Proposition 1. *Any integral manifold $J_h = \{H = h\}$ in T^*E^2 is diffeomorphic to $T_1^*S^2$, i.e., to the bundle of the unit cotangent vectors to the sphere.*

Proof. Let D be the diffeomorphism of E^2 onto $S^2 = \{\xi^2 + \eta^2 + \zeta^2 = 1\}$ such that $D(x, y, z) = (\sqrt{A}x, \sqrt{B}y, \sqrt{C}z)$. The corresponding diffeomorphism of the cotangent bundles $T^*D : T^*S^2 \rightarrow T^*E^2$ has the form $x = \xi/\sqrt{A}$, $y = \eta/\sqrt{B}$, $z = \zeta/\sqrt{C}$, $p_x = \sqrt{A}p_\xi$, $p_y = \sqrt{B}p_\eta$, $p_z = \sqrt{C}p_\zeta$. In \mathbb{R}^6 , the manifold $M^3 = (T^*D)^{-1}(J_h) \subset T^*S^2$ is given by the equations

$$\frac{A^2x^2 + B^2y^2 + C^2z^2}{2h} (Ap_\xi^2 + Bp_\eta^2 + Cp_\zeta^2) = 1, \quad \xi^2 + \eta^2 + \zeta^2 = 1, \quad \xi p_\xi + \eta p_\eta + \zeta p_\zeta = 0.$$

In turn, $T_1^*S^2$ is given by the equations

$$p_\xi^2 + p_\eta^2 + p_\zeta^2 = 1, \quad \xi^2 + \eta^2 + \zeta^2 = 1, \quad \xi p_\xi + \eta p_\eta + \zeta p_\zeta = 0.$$

Let us define $\vartheta : M^3 \rightarrow T_1^*S^2$ by putting $\vartheta(\xi, \eta, \zeta, p_\xi, p_\eta, p_\zeta) = (\xi, \eta, \zeta, p_\xi/\delta, p_\eta/\delta, p_\zeta/\delta)$, where $\delta^2 = p_\xi^2 + p_\eta^2 + p_\zeta^2$. The map ϑ is bijective and $\text{rang } \vartheta = 3$ at each point of M^3 . Therefore ϑ is a diffeomorphism. The composition $\alpha = \vartheta \circ T^*D^{-1}$ takes J_h to T_1^*S . This proves the statement. \square

Let us point out one more property of manifolds J_h . Let Q be the closed ball in \mathbb{R}^3 of radius π with the center at the coordinates origin. Declare the diametrically opposite points of the ball boundary equivalent and denote by P the quotient space of the topological space Q with respect to this equivalence. For each $\nu \in P$, we denote by $v_\nu \in SO(3)$ the element for which ν is the defining vector (see [1]). Let $\omega_0 \in T_1^*S^2$ have the coordinates $\xi = 1$, $\eta = \zeta = 0$, $p_\eta = 1$, $p_\xi = p_\zeta = 0$. The map $\beta : P \rightarrow J_h$ defined as $\beta(\nu) = (v_\nu \circ \alpha)^{-1}(\omega_0)$ is a homeomorphism. We use the map β for a geometric interpretation.

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Let λ, μ be the elliptic coordinates on E^2

$$x^2 = a \frac{(a-\lambda)(a-\mu)}{(a-b)(a-c)}, \quad y^2 = b \frac{(\lambda-b)(b-\mu)}{(a-b)(b-c)}, \quad z^2 = c \frac{(\lambda-c)(\mu-c)}{(a-c)(b-c)},$$

where $a = 1/A, b = 1/B, c = 1/C$. The elliptic coordinates change in the regions $a \geq \lambda \geq b \geq \mu \geq c$. Denote $F(t) = (a-t)(b-t)(c-t)/t$. The Hamilton function (1) takes the form

$$H = 2 \frac{\lambda\mu}{\lambda-\mu} [F(\lambda)p_\lambda^2 - F(\mu)p_\mu^2].$$

Let us introduce on E^2 the Liouville coordinates by the formulas

$$u = \int_b^\lambda \frac{dt}{\sqrt{F(t)}}, \quad v = \int_c^\mu \frac{dt}{\sqrt{-F(t)}}.$$

For them, the regions are

$$0 \leq u \leq m = \int_b^a \frac{dt}{\sqrt{F(t)}}, \quad 0 \leq v \leq n = \int_c^b \frac{dt}{\sqrt{-F(t)}}.$$

In Fig. 1, we show parametric curves of u and v on the ellipsoid. In the coordinates (u, v) ,

$$H = 2[V(v) - U(u)]^{-1}(p_u^2 + p_v^2),$$

where $U(u) = 1/\lambda(u)$, $V(v) = 1/\mu(v)$. Note that $dU/du = -\lambda^{-2}\sqrt{F(\lambda)}$, i.e., $dU/du = 0$ at $u = 0, u = m$ and $dU/du < 0$ at $0 < u < m$. Similarly, $dV/dv = 0$ at $v = 0, v = n$ and $dV/dv < 0$ at $0 < v < n$.

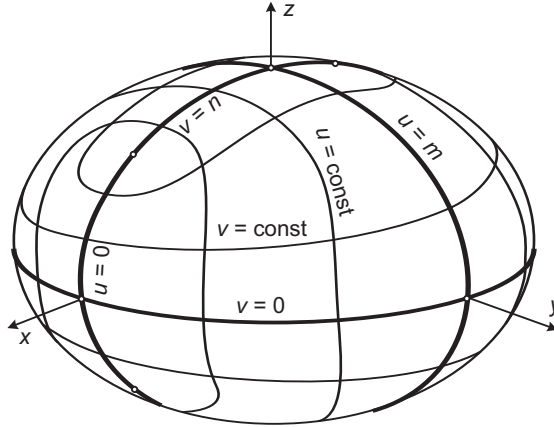


Figure 1: Coordinates on the ellipsoid.

In the domain where u and v are local coordinates the restriction of the initial system to the manifold J_h admits the integrals

$$p_u^2 + hU(u) = h\kappa, \quad p_v^2 - hV(v) = -h\kappa. \quad (2)$$

Denote by $J_{h,\kappa}$ the subset of J_h defined by equations (2). The admissible values of κ are $A \leq \kappa \leq C$. Let us find out the topological type of the integral manifolds $J_{h,\kappa}$ in the following cases: 1) $A \leq \kappa < B$; 2) $B < \kappa \leq C$; 3) $\kappa = B$.

Let $\mathfrak{W} = E^2 \setminus \{u = 0\}$ and $\mathfrak{S} = E^2 \setminus \{v = n\}$ be the regions on the ellipsoid surface. In them, we introduce the local coordinates $\mathfrak{W} = \{(w, \varphi \bmod 4n)\}$, $\mathfrak{S} = \{(s, \theta \bmod 4m)\}$ similar to cylindrical ones putting

$$w = \begin{cases} u & \text{при } x \leq 0; \\ 2m - u & \text{при } x \geq 0, \end{cases} \quad s = \begin{cases} v & \text{при } z \leq 0; \\ -v & \text{при } z \geq 0, \end{cases}$$

$$\varphi = \begin{cases} v & \text{при } y \geq 0, z \geq 0; \\ 2n - v & \text{при } y \leq 0, z \geq 0; \\ 2n + v & \text{при } y \leq 0, z \leq 0; \\ 4n - v & \text{при } y \geq 0, z \leq 0, \end{cases} \quad \theta = \begin{cases} u & \text{при } x \geq 0, y \geq 0; \\ 2m - u & \text{при } x \leq 0, y \geq 0; \\ 2m + u & \text{при } x \leq 0, y \leq 0; \\ 4m - u & \text{при } x \geq 0, y \leq 0. \end{cases}$$

It is easily shown that these coordinates are compatible with the smooth structure of the ellipsoid.

Let us consider the cases 1 – 3.

If $A \leq \kappa < B$, then the motion takes place in the region \mathfrak{W} and the equations admit the first integrals

$$H_w = p_w^2 + hW(w) = h\kappa, \quad H_\varphi = p_\varphi^2 - h\Phi(\varphi) = -h\kappa, \quad (3)$$

where $W(w) = U(u(w))$, $\Phi(\varphi) = V(v(\varphi))$. The qualitative picture of the functions W and Φ is shown in Fig. 2. In Fig. 3, we show the phase portraits of one-dimensional systems corresponding to the Hamilton functions H_w and H_φ . Each manifold $J_{h,\kappa}$ is the product of level lines of the functions H_w and H_φ defined by (3). Thus, $J_{h,A}$ is two non-intersecting circles (they correspond to the cross section of the ellipsoid by the plane $x = 0$ with two different directions of motion). If $A < \kappa < B$, then $J_{h,\kappa}$ consists of two two-dimensional tori each of which concentrically envelopes one of the circles out of $J_{h,A}$.

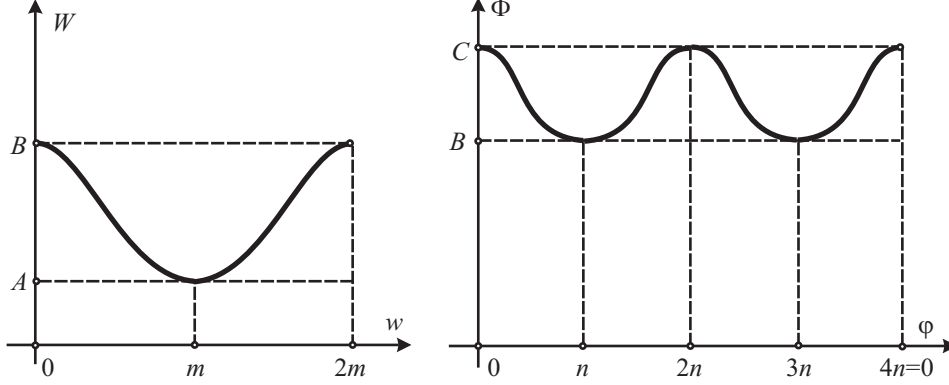


Figure 2: The functions W and Φ .

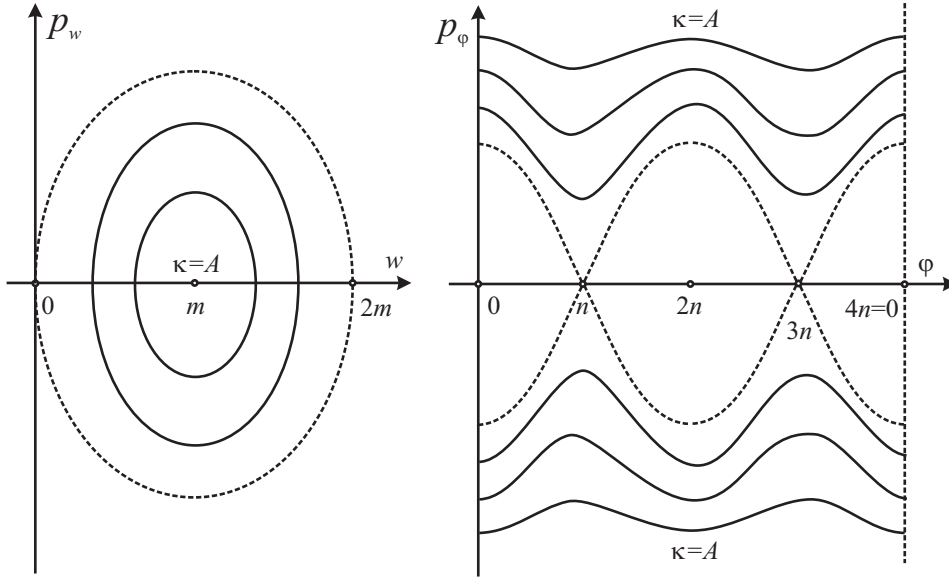


Figure 3: The portraits of one-dimensional systems.

In the case $B < \kappa \leq C$ the motion takes place in the region \mathfrak{S} . In $T^*\mathfrak{S} \cap J_h$ the integrals are defined

$$H_s = p_s^2 - hS(s) = -h\kappa, \quad H_\theta = p_\theta^2 + h\Theta(\theta) = h\kappa,$$

where $S(s) = V(v(s))$, $\Theta(\theta) = U(u(\theta))$; the system splits into two one-dimensional ones. The manifold $J_{h,C}$ consists of two non-intersecting circles and $J_{h,\kappa}$ for $B < \kappa < C$ consists of two two-dimensional tori each of which concentrically envelopes one of the circles out of $J_{h,C}$.

In Fig. 4, where the diametrically opposite points of the ball boundary are identified, we show the sets corresponding to the manifolds $J_{h,A}$ and $J_{h,C}$ under the homeomorphism $\beta : P \rightarrow J_h$. The union of the circles 1 and 2 is the set $\beta^{-1}(J_{h,C})$. The set $\beta^{-1}(J_{h,A})$ consists of the circles 3 and 4.

Now let us consider the case $\kappa = B$. We denote by K_1 , K_2 , K_3 , and K_4 the umbilical points ($u = 0, v = n$) on the ellipsoid surface lying respectively in the regions $\{x > 0, z > 0\}$, $\{x < 0, z > 0\}$, $\{x < 0, z < 0\}$, and $\{x > 0, z < 0\}$.

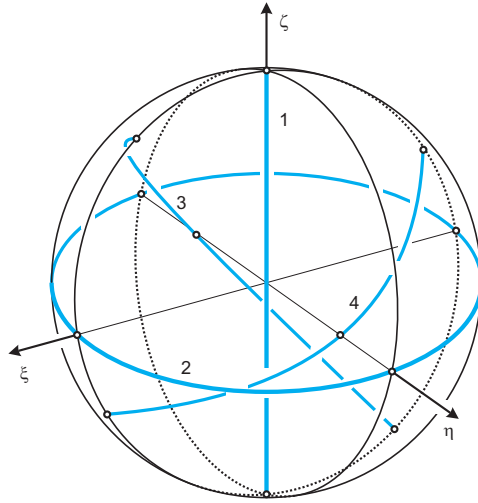


Figure 4: The integral circles.

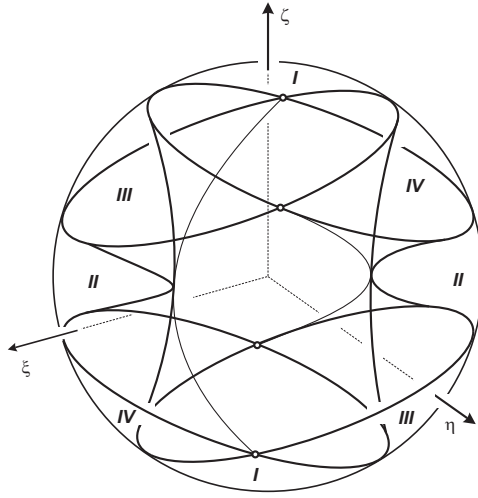


Figure 5: The tori regions.

Proposition 2. *The cross section of the ellipsoid by the plane $y = 0$ is a closed geodesic of the metric $d\Sigma$. All geodesics starting from an umbilical point at $t = 0$ meet simultaneously at the opposite umbilical point.*

Proof. Let us use the coordinates (w, φ) . Introducing the “reduced time” τ by the formula $d\tau = [\Phi(\varphi) - W(w)]^{-1}dt$ and using equations (3) with $\kappa = B$, we get the equations of geodesics in the form

$$\frac{dw}{d\tau} = \pm \sqrt{h(B - W(w))}, \quad \frac{d\varphi}{d\tau} = \pm \sqrt{h(\Phi(\varphi) - B)}. \quad (4)$$

Denote

$$F(w, w_0) = \int_{w_0}^w \frac{dw}{\sqrt{h(B - W(w))}}, \quad G(\varphi, \varphi_0) = \int_{\varphi_0}^{\varphi} \frac{d\varphi}{\sqrt{h(\Phi(\varphi) - B)}}.$$

Let $w = f(\tau, w_0)$ and $\varphi = g(\tau, \varphi_0)$ be the inverse for the dependencies $\tau = F(w, w_0)$ and $\tau = G(\varphi, \varphi_0)$ respectively. Equations (4) admit the solutions

$$\begin{aligned} (w \equiv 0, \varphi = g(\pm\tau, \varphi_0)), & \quad (w \equiv 2m, \varphi = g(\pm\tau, \varphi_0)), \\ (w = f(\pm\tau, w_0), \varphi \equiv n), & \quad (w = f(\pm\tau, w_0), \varphi \equiv 3n). \end{aligned}$$

This proves the first statement.

Consider an arbitrary trajectory of equations (4) starting at a point $\{w_0, \varphi_0\}$ not belonging to the cross section $y = 0$. Let, for definition, this point lie in the first octant, i.e., $m < w_0 < 2m$, $0 < \varphi_0 < n$. The initial velocity may have four directions according to the choice of the signs in (4). Suppose, for example, that $dw/d\tau|_{\tau=0} > 0$, $d\varphi/d\tau|_{\tau=0} > 0$. Then (see Fig. 3) as $\tau \rightarrow +\infty$, the coordinates w and φ monotonously increase and $w \rightarrow 2m$, $\varphi \rightarrow n \pmod{4n}$. As $\tau \rightarrow -\infty$ we have monotonous decreasing $w \rightarrow 0$ and $\varphi \rightarrow -n \pmod{4n}$.

Therefore the chosen trajectory of (4) asymptotically approaches K_1 as $\tau \rightarrow +\infty$ and K_3 as $\tau \rightarrow -\infty$. Another possible cases of the inial directions are considered analogously.

So, since the geodesics starting at an umbilical point can correspond only to the value $\kappa = B$, each such geodesic meets the cross section $y = 0$ for the first time at the opposite umbilical point.

Let $\gamma_1(t)$ and $\gamma_2(t)$ be two geodesics such that $\gamma_1(0) = \gamma_2(0) = K_3$. Suppose that some time value $t = t_0 > 0$ corresponds to the value $\tau = 0$ of the “reduced time”. Let $\gamma_1(t_0) = (w_1, \varphi_1)$, $\gamma_2(t_0) = (w_2, \varphi_2)$. Then the dependency of γ_1 on the “reduced time” is $w = f(\tau, w_1)$, $\varphi = g(\tau, \varphi_1)$, and the equations of γ_2 are $w = f(\tau, w_2)$, $\varphi = g(\tau, \varphi_2)$. Denote by t_1 and t_2 the minimal positive values of t for which $\gamma_1(t_1) = \gamma_2(t_2) = K_1$. Then

$$t_1 = \int_{-\infty}^{+\infty} [\Phi(g(\tau, \varphi_1)) - W(f(\tau, \varphi_1))] d\tau, \quad (5)$$

$$t_2 = \int_{-\infty}^{+\infty} [\Phi(g(\tau, \varphi_2)) - W(f(\tau, \varphi_2))] d\tau. \quad (6)$$

The integrals in (5) and (6) converge since the metric $d\Sigma$ does not have singularities.

Let us show that $t_1 = t_2$. For this purpose we use the obvious relations

$$f(\tau, w_1) = f(\tau - F(w_1, w_2), w_2), \quad g(\tau, w_1) = g(\tau - G(w_1, w_2), w_2) \quad (7)$$

and the following almost obvious statement. Suppose that for a function $\psi(\tau)$ ($-\infty < \tau < +\infty$) there exists such a point τ_0 that $\chi(\tau) = \psi(\tau + \tau_0)$ is an even function. If the integral

$$\int_{-\infty}^{+\infty} [\psi(\tau) - \psi(\tau + k)] d\tau,$$

with some constant k converges, then it equals zero. Using (7), we transform (5) as follows

$$t_1 = \int_{-\infty}^{+\infty} [\Phi(g(\tau, \varphi_2)) - W(f(\tau + G(\varphi_2, \varphi_1)) - F(w_2, w_1), w_2)] d\tau.$$

Then we subtract the equality (6):

$$t_1 - t_2 = \int_{-\infty}^{+\infty} [W(f(\tau, w_2)) - W(f(\tau + k, w_2))] d\tau.$$

Here

$$k = G(\varphi_2, \varphi_1) - F(w_2, w_1)$$

does not depend on τ .

It is easy to check that $W(f(\tau, w_2))$ as a function of τ satisfies the condition of the just formulated statement. For this, it is sufficient to choose τ_0 in such a way that $f(\tau_0, w_2) = m$. Consequently, $t_1 = t_2$. The proposition is proved. \square

Let us now describe the type of the set $J_{h,B}$. The curves $O_i = J_h \cap T_K^* E^2$ ($i = 1, 2, 3, 4$) are topological circles. According to Proposition 2, all trajectories starting at O_1 simultaneously cross O_3 and simultaneously return to O_1 . Therefore this family of trajectories fills a closed flow tube, i.e., they fill a two-dimensional torus T_1 in J_h . In the same way the family of geodesics crossing K_2 and K_4 fills a two-dimensional torus T_2 in J_h . The tori T_1 and T_2 intersect by two circles corresponding to the cross section of the ellipsoid by the plane $y = 0$ with two deffrent directions of motion.

In Fig. 5, we show how the set $\beta^{-1}(J_{h,B})$ is embedded in P (the diametrically opposite points of the ball boundary are identified). The regions $I - IV$ are filled with the one-parameter families of the integral tori enveloping concentrically the circles $1 - 4$ respectively (see Fig. 4).

References

- [1] *Kharlamov M.P.* Reduction in mechanical systems with symmetry // *Mekh. Tverd. Tela.* – 1976. – N 8. – P. 4–18. [arXiv:1401.4393](#).